On the Realizability of Electric Fields in Conducting Materials

Marc Briane[1] Graeme Milton[2] Andrejs Treibergs[2]*

[1]Institut de Recherche Mathématique de Rennes, INSA de Rennes

[2]Department of Mathematics, University of Utah

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"Which electric fields are realizeable in conducting materials?"

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3. Outline.

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- Let $Y = [0,1]^d$ be the unit cube in $\mathbf{R}^d$ and $\sigma \in \mathcal{L}^\infty(\mathbf{R}^d, \mathbf{R}^{d\times d})$ be symmetric, uniformly elliptic conductivity. Assume σ is Y-periodic:

$$\sigma(x+k) = \sigma(x) \qquad \text{for all } x \in \mathbf{R}^d \text{ and } k \in \mathbf{Z}^d.$$

  For all $\lambda \in \mathbf{R}^d - \{0\}$ there is $u^\lambda \in \mathcal{H}^1_{\text{loc}}(\mathbf{R}^d)$, unique up to constant multiple, such that $u(x) - \lambda \bullet x$ is Y-periodic and

$$\operatorname{div}(\sigma \nabla u^\lambda) = 0 \tag{1}$$

  The effective conductivity of the periodic material is then σ^* given by averaging over a cell

$$\sigma^* \lambda = \sigma^* \langle \nabla u^\lambda \rangle = \langle \sigma \nabla u^\lambda \rangle$$

  where ∇u^λ is the electric field and $J = \sigma \nabla u^\lambda$ is the current field.

- We reverse the question: given a periodic electric field ∇u, is it possible to find a symmetric periodic positive definite conductivity σ that satisfies the conductivity equation (1)? In other words, which electric fields are realizable?

Consider the case that conductivity $\sigma = sI$ is isotropic.

Theorem (𝕀 1)

Assume that $u \in \mathcal{C}^2(\mathbf{R}^d)$ satisfies $\nabla u \neq 0$. Then ∇u is locally isotropically realizable.

Let $x_0 \in D$. Writing $\sigma = e^z$ the conductivity equation $\operatorname{div}(e^z \nabla u) = 0$ becomes a first order PDE for the unknown $z(x)$,

$$\nabla u(x) \bullet \nabla z = -\Delta u(x) \tag{2}$$

The usual method of characteristics gives the solution. Since ∇u is a characteristic direction, if $\mathcal{H}$ is a hypersurface through x_0, transverse to $\nabla u(x_0)$ and $z_0(h)$ a function on $\mathcal{H}$, then the solution may be given by using the PDE to propagate the solution off of $\mathcal{H}$. Let $X(t,x)$ be the gradient flow of ∇u, satisfying the characteristic ODE for $(t,h) \in I \times G$ in some neighborhood G of x_0 and some $I = (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$,

$$\begin{aligned} \frac{\partial}{\partial t} X(t,h) &= \nabla u\big(X(t,h)\big), \qquad \text{for } (t,h) \in I \times G \\ X(0,h) &= h \end{aligned}$$

Then z satisfies an ODE along the trajectories since

$$\frac{\partial}{\partial t} z\big(X(t,h)\big) = \nabla z\big(X(t,h)\big) \bullet \frac{\partial}{\partial t} X(t,h) = -\,\Delta u\big(X(t,h)\big)$$

If also the initial condition holds

$$z(0,h) = z_0(h) \qquad \text{for } h \in G \cap \mathcal{H}$$

then the solution is

$$\zeta(t,h) = z_0(h) - \int_0^t \Delta u\big(X(\tau,h)\big)\, d\tau$$

Finally, the mapping $\Psi : (t,h) \mapsto X(t,h)$ is a local $\mathcal{C}^1$ diffeomorphism from $I \times (G \cap \mathcal{H})$ to a neighborhood of x_0 since the Jacobian $d\Psi(0,x_0)$ is invertible because $\nabla u(x_0)$ is transverse to $\mathcal{H}$. Writing its inverse $(t,h) = \Phi(x)$, a solution of (2) near x_0 is

$$z(x) = \zeta\big(\Phi(x)\big).$$

We say that the hypersurface $\mathcal{H}$ is a global section for the flow of ∇u if the trajectory of the gradient flow starting from any point $y \in \mathbf{R}^d$ meets $\mathcal{H}$ transversally in exactly one point.

Theorem ($\mathbb{I}$ 2)

Assume that $u \in \mathcal{C}^2(\mathbf{R}^d)$ satisfies $\nabla u \neq 0$ and that ∇u has a global section $\mathcal{H}$. Then ∇u is isotropically globally realizable.

Note that if ∇u is periodic then z may not be periodic.

Example (1)

Let $u(x, y) = x - \cos(2\pi y)$, and $Y = [0, a] \times [0, 1]$, where $a > 0$.

$$\nabla u = \mathbf{e}_1 + 2\pi \sin(2\pi y)\mathbf{e}_2, \qquad \Delta u = 4\pi^2 \cos(2\pi y).$$

On the section $x = x_1$ the initial condition is $X(0, x) = x$ and the gradient flow decouples

$$\begin{aligned} \tfrac{\partial}{\partial t} X_1 &= 1 \\ \tfrac{\partial}{\partial t} X_2 &= 2\pi \sin\big(2\pi X_2(t, x)\big) \end{aligned}$$

It can be integrated: for $x = (x_1, x_2)$

$$X(t, x) = \left(x_1 + t, n + \frac{1}{\pi} \arctan\left(e^{4\pi^2 t} tan(\pi x_2)\right)\right), \text{ if } x_2 \in (n - \frac{1}{2}, n + \frac{1}{2})$$

$$X(t, x) = \left(x_1 + t, n + \frac{1}{2}\right), \qquad \text{if } x_2 = n + \frac{1}{2}$$

Also

$$\frac{\partial}{\partial t} z = \nabla z \bullet \frac{\partial}{\partial t} X = -\Delta u\big(X(t,x)\big) = \frac{4\pi^2 e^{8\pi^2 t} tan^2(\pi x_2) - 4\pi^2}{e^{8\pi^2 t} tan^2(\pi x_2) + 1}$$

If z vanishes at $x_1 = 0$, this can be integrated to yield

$$\sigma = e^z = \begin{cases} \dfrac{1 + \tan^2(\pi x_2)}{e^{4\pi^2 x_1} + e^{-4\pi^2 x_1} \tan^2(\pi x_2)}, & \text{if } x_2 \notin \frac{1}{2} + \mathbf{Z}; \\ e^{4\pi^2 x_1}, & \text{if } x_2 \in \frac{1}{2} + \mathbf{Z};. \end{cases} \tag{3}$$

We see it is not periodic in Y.

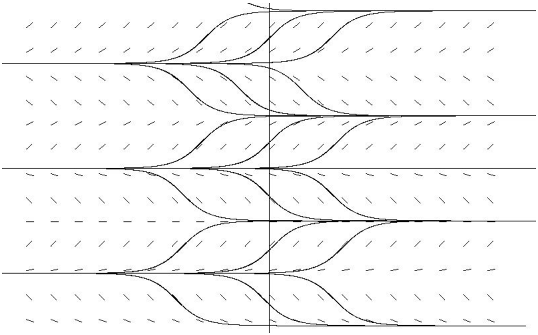


Figure: Trajectories of the Gradient Flow for Example 1.

Example (2)

Let the characteristic function of periodic intervals be given by $\chi(t) = 1$ if $0 \leq \lfloor t \rfloor \leq \frac{1}{2}$ (fractional part) and 0 otherwise. Then

$$u(x, y) = y - x + \int_0^x \chi(t)\, dt$$

is Lipschitz continuous and

$$\nabla u = \chi \mathbf{e}_2 + (1 - \chi)(\mathbf{e}_2 - \mathbf{e}_1) \qquad a.e. \text{ in } \mathbf{R}^2,$$

For this ∇u there is no positive function $\sigma \in \mathcal{L}^\infty(\mathbf{R}^2)$ such that $\sigma \nabla u$ is divergence free.

∇u has discontinuities on the lines $x_1 = k/2$ for some $k \in \mathbf{Z}$. Let $Q = (-r, r)^2$ for some $r \in (0, \frac{1}{2})$. If there were positive $\sigma \in \mathcal{L}^\infty(Q)$ such that $\sigma \nabla u$ is divergence free, then there is a stream function $v \in \mathcal{H}^1$ satisfying $\nabla v = R\sigma \nabla u$, which is unique up to additive constant and is Lipschitz continuous.

$\nabla v = R\sigma\nabla u$ implies

$$0 = \nabla u \bullet \nabla v = (\mathbf{e}_2 - \mathbf{e}_1) \bullet \nabla v \qquad \text{in } (-r, 0) \times (-r, r)$$

hence $v(x, y) = f(x + y)$ for some Lipschitz function f in $[-2r, r]$. On the other hand

$$0 = \nabla u \bullet \nabla v = \mathbf{e}_2 \bullet \nabla v \qquad \text{in } (0, r) \times (-r, r)$$

Hence $v(x, y) = g(x)$ for some Lipschitz function g in $[0, r]$.
By continuiuty on the line $x_1 = 0$, $f(y) = g(0)$. Hence f is constant on $[-r, r]$ implying v is too. Thus

$$\begin{aligned} \nabla v &= 0 \ a.e. && \text{in } (-r, 0) \times (0, r) \text{ and} \\ \sigma\nabla u &= \sigma(\mathbf{e}_2 - \mathbf{e}_1) \neq 0 \ a.e. && \text{in } (-r, 0) \times (0, r) \end{aligned}$$

which contradicts the equality$\nabla v = R\sigma\nabla u$ *a.e.* Thus ∇u is not isotropically realizable in neighborhoods near the lines $x = k/2$, $k \in \mathbf{Z}$. □

Theorem ($\mathbb{I}$ 3)

Let $Y \in \mathbf{R}^d$ be a closed parallelepiped. Assume that $u \in \mathcal{C}^1(\mathbf{R}^d)$ satisfies

- *∇u is Y-periodic and the cell average $\langle \nabla u \rangle \neq 0$.*
- *∇u is realized as an electric field associated with a smooth periodic conductivity.*

Then

1. *if $d = 2$ then $\nabla u \neq 0$ in all of $\mathbf{R}^2$;*
2. *if $d = 3$ then there is an example where $\nabla u(y_0) = 0$ for some point $y_0 \in \mathbf{R}^3$.*

(1) Follows from a theorem of Allesandrini & Nesi[2001] about solutions of regular elliptic equations.
(2) One example is given by Ancona[2002], another may be constructed from the periodic chain mail of Briane, Milton and Nesi[2004].

Theorem (Alessandrini & Nesi (2001))

Let $Y \subset \mathbf{R}^2$ *be a parallelogram,* $\sigma \in \mathcal{L}^\infty$ *be uniformly positive definite, symmetric and* Y*-periodic. For a symmetric matrix* A *with* $\det A > 0$ *consider* $U \in \mathcal{W}^{2,2}_{loc}(\omega, \mathbf{R}^2)$ *such that* $U - Ax$ *is a* Y*-periodic and satisfies*

$$\mathrm{Div}(\sigma DU) = 0$$

and the cell average $\langle \det(DU) \rangle > 0$. *Then*

$$\det(DU) > 0 \qquad a.\ e.\ in\ \mathbf{R}^2.$$

In the isotropic case u *is a scalar,* $\langle \nabla u \rangle \neq 0$ *impies* $\nabla u \neq 0$ *in* $\mathbf{R}^2$.

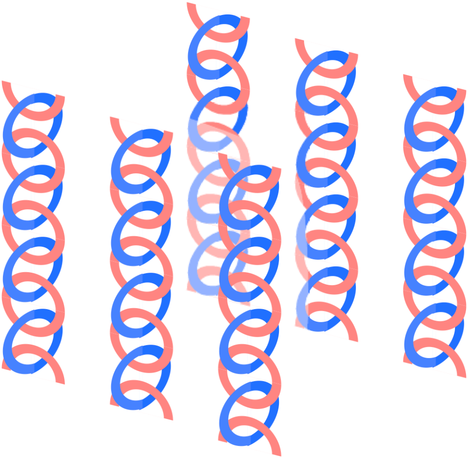
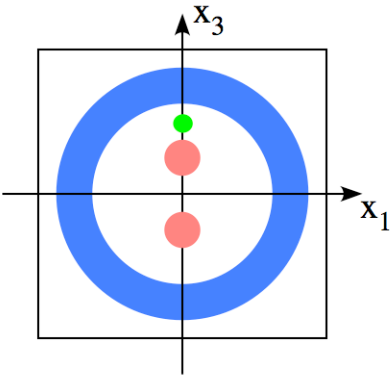


Figure: Periodic chain mail of Briane, Milton and Nesi consisting of linked toroidal rings of highly conductive material.



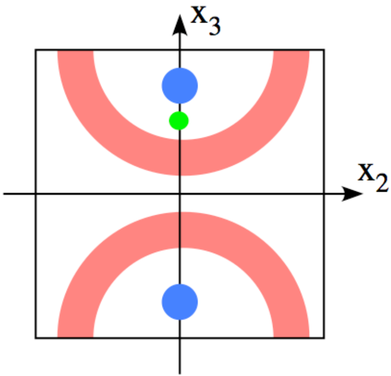


Figure: Section of periodic chain mail.

Rings have $\sigma \gg 1$. There is a matrix field such that $\langle DU \rangle = I$, $\langle \det(DU) \rangle = 1$ but $\det(DU) < 0$ in green region.
Hence there is $\lambda \in \mathbf{R}^3 - \{0\}$ such that $\nabla(u \bullet \lambda)$ vanishes in $\mathbf{R}^3$.

Theorem ($\mathbb{I}$ 4)

Let $Y \subset \mathbf{R}^d$ be a compact parallelopiped and $d \geq 2$. Let $u \in \mathcal{C}^3(\mathbf{R}^d)$ such that ∇u is Y-periodic,

$$\nabla u \neq 0 \textit{ in } \mathbf{R}^d \textit{ and the cell average } \langle \nabla u \rangle \neq 0.$$

Then ∇u is globally isotropically realizable.

Since ∇u is nonvanishing and periodic, $0 < c_1 \leq |\nabla u(x)| \leq c_2$ for all x and the function $f(t) = u(X(t, x_0))$ satisfies

$$f'(t) = \nabla u(X(t, x-0)) \bullet \frac{\partial X}{\partial t}(t, x_0) = |\nabla u(X(t, x_0))|^2 \in \left[c_1^2, c_2^2\right].$$

Thus

$$\lim_{t \to \infty} f(t) = \infty \text{ and } \lim_{t \to -\infty} f(t) = -\infty$$

and there is a unique $\tau(x) \in \mathbf{R}$ such that $f(\tau(x)) = 0$. By differentiable dependence and the implicit function theorem $\tau \in \mathcal{C}^2(\mathbf{R}^d)$.

Hence the level set $\{x \in \mathbf{R}^d : \tau(x) = 0\}$ is a $\mathcal{C}^1$ global section. Put

$$w(x) = \int_0^{\tau(x)} \Delta u\big(X(s,x)\big)\, ds \qquad \text{for } x \in \mathbf{R}^d$$

By the change of variables formula $r = s + t$,

$$w\big(X(t,x)\big) = \int_0^{\tau(x)-t} \Delta u\big(X(s+t,x)\big)\, ds = \int_t^{\tau(x)} \Delta u\big(X(r,x)\big)\, dr$$

so

$$\frac{\partial}{\partial t} w\big(X(t,x)\big) = \nabla w\big(X(t,x)\big) \bullet \nabla u\big(X(t,x)\big) = -\,\Delta u\big(X(t,x)\big)$$

For the conductivity $\sigma = e^{w(x)}$ we have at $t = 0$

$$\operatorname{div}\big(\sigma \nabla(u)\big) = e^{w}\big(\nabla w \bullet \nabla v + \Delta u\big) = 0. \quad \square$$

Theorem (𝕀5. For Example 1, no periodic isotropic σ is possible.)

For $u(x,y) = x - \cos(2\pi y)$, the $Y = [0,a] \times [0,1]$-periodic electric field ∇u does not admit a continuous non-vanishing Y-periodic isotropic conductivity σ that makes $\sigma \nabla u$ divergence free.

Note $\nabla u = \mathbf{e}_1 + 2\pi \sin(2\pi y)\mathbf{e}_2$. Assume there is a Y-periodic function σ such that $\sigma \nabla u$ is divergence free. Let $Q = [0,a] \times [-r,r]$ for some $0 < r < \frac{1}{2}$. Then, using Green's Theorem,

$$
\begin{aligned}
0 = \int_Q \operatorname{div}(\sigma \nabla u)\, dx\, dy &= \oint_{\partial Q} (\sigma u_x)\, dy - (\sigma u_y)\, dx \\
&= \int_{-r}^{r} \big[\sigma(a,y) u_x(a,y) - \sigma(0,y) u_x(0,y)\big]\, dy \\
&\quad + \int_0^a \big[\sigma(x,r) u_y(x,r) - \sigma(x,-r) u_y(x,-r)\big]\, dx \\
&= 0 + 2\pi \sin(2\pi r) \int_0^a \big[\sigma(x,r) + \sigma(x,-r)\big]\, dx > 0 \quad \square
\end{aligned}
$$

Theorem ($\mathbb{I}$ 6.)

Let $Y \subset \mathbf{R}^d$ be a compact parallelopiped and $d \geq 2$. Let $u \in \mathcal{C}^3(\mathbf{R}^d)$ such that ∇u is Y-periodic, $\nabla u \neq 0$ in $\mathbf{R}^d$ and the cell average $\langle \nabla u \rangle \neq 0$. Assume that there is $C < \infty$ such that for all $x \in \mathbf{R}^d$,

$$\left| \int_0^{\tau(x)} \Delta u\big(X(t,x)\big)\, dt \right| \leq C \tag{4}$$

where $\tau(x)$ is the unique time such that $u(\tau(x), x) = 0$ as in the proof of Theorem $\mathbb{I}$ 4. Then ∇u is isotropically realizable with Y-periodic conductivity $\sigma, \sigma^{-1} \in \mathcal{L}^\infty_Y(\mathbf{R}^d)$.

Conversely, if ∇u is isotropically realizable with Y-periodic conductivity $\sigma \in \mathcal{C}^1_Y(\mathbf{R}^d)$, then (4) holds.

Example (1, cont. Assumptions of Theorem 𝕀6 do not hold.)

Let $u(x, y) = x - \cos(2\pi y)$, and $Y = [0, a] \times [0, 1]$, where $a > 0$.

$$\nabla u = \big(1, 2\pi \sin(2\pi x_2)\big)$$
$$\Delta u = 4\pi^2 \cos(2\pi x_2)$$

Put $p_0 = (x_1, 0)$. Thus, $X(t, p_0) = (x_1 + t, 0)$ so

$$w(p_0) = \int_0^{\tau(p_0)} \Delta u\big(X(t, p_0)\big)\, dt = 4\pi^2 \cos(0)\tau(p_0).$$

But by the definition of τ,

$$0 = u\big(X(\tau(p_0), p_0)\big) = x_1 + \tau(p_0) - \cos(0)$$

so that

$$w(p_0) = 4\pi^2(1 - x_1)$$

which is not bounded.

Assume there is a positive periodic $\sigma = e^w \in \mathcal{C}^1_Y(\mathbf{R}^d)$ such that $\operatorname{div}(\sigma \nabla u) = 0$. Then $\nabla u \bullet \nabla w + \Delta u = 0$ in $\mathbf{R}^d$. Hence

$$\begin{aligned}
\int_0^{\tau(x)} \Delta u\big(X(t,x)\big)\,dt &= -\int_0^{\tau(x)} \nabla w\big(X(t,x)\big) \bullet \nabla u\big(X(t,x)\big)\,dt \\
&= -\int_0^{\tau(x)} \nabla w\big(X(t,x)\big) \bullet \frac{\partial}{\partial t} X(t,x)\,dt \\
&= w\big(X(0,x)\big) - w\big(X(\tau(x),x)\big) \\
&= x - w\big(X(\tau(x),x)\big)
\end{aligned}$$

which is bounded by assumption. Hence (4) follows.

For simplicity, assume $Y = [0,1]^d$. For $x \in \mathbf{R}^d$ define

$$\sigma_0(x) = \exp\left(\int_0^{\tau(x)} \Delta u\big(X(t,x)\big)\, dt\right)$$

and for $n \in \mathbf{N}$, average over the $(2n+1)^d$ integer vectors in $[-n,n]^d$

$$\sigma_n(x) = \frac{1}{(2n+1)^d} \sum_{k \in \mathbf{Z}^d \cap [-n,n]^d} \sigma_0(x+k)$$

By (4), σ_n is bounded in $\mathcal{L}^\infty(\mathbf{R}^d)$. Hence a subsequence $\sigma_{n'}$ converges weak-$*$ to σ_∞ in $\mathcal{L}^\infty(\mathbf{R}^d)$.

For any $k \in \mathbf{Z}^d$

$$\begin{aligned}
&\Big|(2n+1)^d \sigma_n(x+k) - (2n+1)^d \sigma_n(x)\Big| \\
&= \Big| \sum_{|j-k|_\infty \le n} \sigma_n(x+j) - \sum_{|j|_\infty \le n} \sigma_n(x+j) \Big| \\
&\le \sum_{\substack{|j|_\infty \le n+|k|_\infty \\ |j|_\infty > n}} \sigma_n(x+k) + \sum_{\substack{|j-k|_\infty \le n+|k|_\infty \\ |j-k|_\infty > n}} \sigma_n(x+k) \\
&\le 2e^C \left((2n+2k+1)^d - (2n+1)^d \right) \le C_2(C,d,k) n^{d-1}
\end{aligned}$$

Letting $n' \to \infty$ implies that $\sigma_\infty(x+k) = \sigma_\infty(x)$ *a.e.* in $\mathbf{R}^d$ and for any k. Thus $\sigma_\infty \in \mathcal{L}^\infty_Y(\mathbf{R}^d)$. Since σ_0 is bounded below by e^{-C}, $\sigma_\infty^{-1} \in \mathcal{L}^\infty_Y(\mathbf{R}^d)$.

As $\nabla u \in \mathcal{C}^2_Y(\mathbf{R}^d)$, it is realized by the conductivity σ_0. Periodicity implies that also $\mathrm{div}(\sigma_n \nabla u) = 0$ in $\mathbf{R}^d$. From weak-$*$ convegence, for every $\varphi \in \mathcal{C}^\infty_c(\mathbf{R}^d)$ we have

$$0 = \lim_{n' \to \infty} \int_{\mathbf{R}^d} \sigma_{n'} \nabla u \bullet \nabla \varphi \, dx = \int_{\mathbf{R}^d} \sigma_\infty \nabla u \bullet \nabla \varphi \, dx$$

Hence $\mathrm{div}(\sigma_\infty \nabla u) = 0$ in $\mathcal{D}'(\mathbf{R}^d)$ so that ∇u is isotropically realized by the Y-periodic conductivity σ_∞. □

Theorem ($\mathbb{A}$ 1)

Let $Y \subset \mathbf{R}^2$ be a closed parallelogram. Let $u \in \mathcal{C}^1(\mathbf{R}^2)$ such that $\nabla u \neq 0$ is Y-periodic in $\mathbf{R}^2$ and the cell average $\langle \nabla u \rangle \neq 0$. Then necessary and sufficient that ∇u be realizable by a continuous, Y-periodic, symmetric positive definite matrix-valued conductivity σ is that there is a functiom $v \in \mathcal{C}^1(\mathbf{R}^2)$ such that ∇v is Y-periodic in $\mathbf{R}^2$ and the cell average $\langle \nabla v \rangle \neq 0$ such that

$$R\nabla u \bullet \nabla v = \det(\nabla u, \nabla v) > 0 \qquad \textit{everywhere in } \mathbf{R}^2. \tag{5}$$

where R is rotation by a right angle.

Theorem $\mathbb{A}$ 1 continues to hold under the weaker assumptions that ∇u is Y-periodic, $\nabla u \in \mathcal{L}^2(Y)$, $\nabla u \neq 0$ *a.e.* in $\mathbf{R}^2$ and $\langle \nabla u \rangle \neq 0$. In this case, the Y-periodic conductivity σ defined only *a.e.* by the formula below and does not remain bounded in Y. However $\sigma \nabla u$ is divergence free in the sense of distributions.

Assume there is such v. (5) says that ∇v is nonvanishing. Then define

$$\sigma = \frac{1}{|\nabla u|^4} \begin{pmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ -\frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_1} \end{pmatrix}^T \begin{pmatrix} R\nabla u \bullet \nabla v & -\nabla u \bullet \nabla v \\ -\nabla u \bullet \nabla v & \frac{|\nabla u \bullet \nabla v|^2 + 1}{R\nabla u \bullet \nabla v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ -\frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_1} \end{pmatrix}$$

which is a continuous, symmetric positive definite matrix function.
$\sigma \nabla u = -R\nabla v$ in $\mathbf{R}^2$ so it is divergence free.
Now assume there is u and a continuous positive definite symmetric σ.
Let $v \in \mathcal{C}^1(\mathbf{R}^2)$ be the stream function which satisfies $\nabla v = -R\nabla u$.
Hence ∇v is Y-periodic and

$$R\nabla u \bullet \nabla v = -\nabla u \bullet R\nabla v = \sigma \nabla u \bullet \nabla u$$

Allesandrini & Nesi's result implies ∇u is nonvanishing, which implies (5). By the div-curl lemma,

$$\langle R\nabla u \bullet \nabla v \rangle = R\langle \nabla u \rangle \bullet \langle \nabla v \rangle = \langle \sigma \nabla u \bullet \nabla u \rangle > 0$$

so $\langle \nabla v \rangle > 0$ also. □

Example (1)

Let $u(x,y) = x - \cos(2\pi y)$, and $Y = [0,a] \times [0,1]$, where $a > 0$. Then ∇u is anisotropically realizble.

$\nabla u = \mathbf{e}_1 + 2\pi \sin(2\pi x_2)\mathbf{e}_2$. Take

$$v(x) = x_2$$

We find

$$R\nabla u \bullet v = (-2\pi \sin(2\pi x_2)\mathbf{e}_1 + \mathbf{e}_2) \bullet \mathbf{e}_2 = 1$$

so Theorem $\mathbb{A}\,1$ applies: for $\delta = 1 + 4\pi^2 \sin^2(2\pi x_2)$, let

$$\sigma = \frac{1}{\delta^2} \begin{pmatrix} \delta^2 + \delta - 1 & -2\pi \sin(2\pi x_2) \\ -2\pi \sin(2\pi x_2) & 1 \end{pmatrix}$$

Now $\sigma \nabla u = \mathbf{e}_1$ which is divergence free.

Example (2)

$u(x) = x_2 - x_1 + \int_0^{x_1} \chi(t)\,dt$ where $\chi = 1$ if $0 \le \lfloor t \rfloor \le \frac{1}{2}$ and 0 otherwise.
Then ∇u is anisotropically realizable.

$\nabla u = \chi \mathbf{e}_2 + (1-\chi)(\mathbf{e}_2 - \mathbf{e}_1)$ *a.e.* in $\mathbf{R}^2$ satisfies the weaker assumptions. For *a.e.* $x \in \mathbf{R}^2$, define

$$v(x) = -x_2 - \int_0^{x_1} \chi(t)\,dt, \qquad \nabla v = -\chi(\mathbf{e}_1 + \mathbf{e}_2) - (1-\chi)\mathbf{e}_2$$

so that *a.e.* in $\mathbf{R}^2$, $-\nabla u \bullet \nabla v = R\nabla u \bullet \nabla v = 1$.
Then formula (5) yields the rank one laminate conductivity *a.e.* in $\mathbf{R}^2$,

$$\sigma = \chi \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1-\chi}{4} \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence *a.e.* in $\mathbf{R}^2$, $\sigma \nabla u = \chi(-\mathbf{e}_1 + \mathbf{e}_2) + (1-\chi)\mathbf{e}_2$ which is divergence free in $\mathcal{D}'(\mathbf{R}^2)$.

Let $d \geq 2$ and $\Omega \subset \mathbf{R}^d$ be open. If $U \in \mathcal{H}^1(\Omega, \mathbf{R}^d)$ then the matrix electric field DU is said to be realizable if there is a symmetric positive definite matrix-valued function $\sigma \in \mathcal{L}^\infty_{\text{loc}}(\Omega, \mathbf{R}^{d\times d})$ such that

$$\text{Div}(\sigma DU) = 0$$

Theorem ($\mathbb{M}$ 1)

Let $d \geq 2$ and $Y \subset \mathbf{R}^d$ be a closed parallelopiped. Let $U \in \mathcal{C}^1(\mathbf{R}^d, \mathbf{R}^d)$ such that DU is Y-periodic.

1. *Assume also* $\det(\langle DU\rangle DU) > 0$ *in* $\mathbf{R}^d$ *and* $\det(\langle DU\rangle) \neq 0$. *Then DU is a realizable matrix electric field with continuous conductivity.*
2. *If $d = 2$,* $\det(\langle DU\rangle) \neq 0$ *and the matrix electric field realized by a $\mathcal{C}^1$ conductivity, then* $\det(\langle DU\rangle DU) > 0$.
3. *If $d = 3$ there exists a smooth Y-periodic matrix field DU such that* $\det(\langle DU\rangle) \neq 0$ *and an associated smooth periodic conductivity σ such that* $\det(DU)$ *takes both positive and negative values in* $\mathbf{R}^3$.

(*i.*) For Y-periodic $U \in \mathcal{C}^1$ such that $\det(\langle DU \rangle DU) \neq 0$ we define

$$\sigma = \det\big(\langle DU \rangle DU\big)\,(DU^{-1})^T\, DU^{-1} = \det\big(\langle DU \rangle\big)\ \mathrm{Cof}(DU)\, DU^{-1}$$

where Cof is the cofactor matrix. σ is Y-periodic, continuous, symmetric and positive definite. Also by Piola's identity, as a distribution,

$$\mathrm{Div}(\mathrm{Cof}\, DU) = 1 \qquad \text{in } \mathcal{D}'(\mathbf{R}^d)$$

Hence σDU is divergence free and DU is realizable with associated conductivity σ.

(*ii.*) Follows from a theorem of Alessandrini and Nesi.
(*iii.*) Example is constructed from periodic chain mail constructed by Briane, Milton and Nesi.

The result *(i.) may be generalized:*

Corollary ($\mathbb{M}$ 2)

Let $d \geq 2$ and $Y \subset \mathbf{R}^2$ be a closed parallelopiped. Let $U \in \mathcal{C}^1(\mathbf{R}^d, \mathbf{R}^d)$ with Y-periodic DU, $\det(\langle DU \rangle DU) > 0$ in $\mathbf{R}^d$ and $\det(\langle DU \rangle) \neq 0$. Then the matrix electric field DU is realized by a family of continuous conductivities σ_φ parameterized by convex functions $\varphi \in \mathcal{C}^2(\mathbf{R}^d)$, those whose Hessian matrices $D^2\varphi$ are positive definite everywhere in $\mathbf{R}^d$.

Define

$$\sigma_\varphi = \det\big(\langle DU \rangle\big)\ \mathrm{Cof}\big(D(\nabla\varphi \circ U)\big)\, DU^{-1}$$

$\sigma_\varphi DU$ is divergence free by Piola's identity. We also have

$$\mathrm{Cof}\big(D(\nabla\varphi \circ U)\big) = \mathrm{Cof}\big(DU\, D^2\varphi \circ U\big) = \mathrm{Cof}\big(DU\big)\ \mathrm{Cof}\big(D^2\varphi \circ U\big)$$

so that σ_φ satisfies

$$\sigma_\varphi = \det\big(\langle DU \rangle DU\big)\, \big(DU^{-1}\big)^T\ \mathrm{Cof}\big(D^2\varphi \circ U\big)\, DU^{-1}$$

Since $D^2\varphi$ is symmetric positive definite, so is its cofactor matrix. Thus σ_φ is an admissible, continuous with $\sigma_\varphi DU$ divergence free in $\mathbf{R}^d$. □

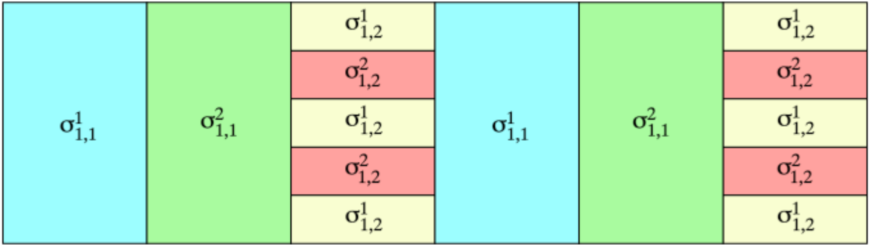


Figure: A rank-two laminate with directions $\xi_1 = \mathbf{e}_1$ and $\xi_{1,2} = \mathbf{e}_2$.

Let $d, n \in \mathbf{N}$. A rank-n laminate in $\mathbf{R}^d$ is a multiscale microstructure defined at n ordered scales $\varepsilon_n \ll \cdots \ll \varepsilon_1$ depending on a small positive parameter $\varepsilon \to 0$ and in multiple directions in $\mathbf{R}^d \setminus \{0\}$, by the following process.

- At the smallest scale ε_n there is a set of m_n rank-one laminates, the ith one of which, for $i = 1, \ldots, m_n$, is composed of an ε_n periodic repetition in the $\xi_{i,n}$ direction of homogeneous layers with constant positive definite conductivity matrices $\sigma^h_{i,n}$, $h \in I_{i,n}$.
- At the scale ε_k there is a set of m_k laminates, the ith one of which, for $i = 1, \ldots, m_k$, is composed of an ε_k-periodic repetition in the $\xi_{i,k} \in \mathbf{R}^d \backslash \{0\}$ direction of homogeneous layers and/or a slection of the m_{k+1} laminates which are obtained at stage $k + 1$ with constant positive definite conductivity matrices $\sigma^h_{i,k}$ and/or $\sigma^h_{i,j}$, resp., for $j = k + 1, \ldots, n$, $h \in I_{i,j}$.
- At the scale ε_1 there is a single laminate ($m_1 = 1$) which is composed of an ε_1-periodic repetition in the $\xi_1 \in \mathbf{R}^d \backslash \{0\}$ direction of homogeneous layers and/or a selection of the m_2 laminates which are obtained at scale ε_2 with constant positive definite conductivity matrices $\sigma^h_{i,1}$ and/or $\sigma^h_{i,j}$, resp., for $j = 2, \ldots, n$, $h \in I_{i,j}$.

The laminate conductivity at stage $k = 1, \ldots, n$ is denoted by $L^\varepsilon_k(\hat{\sigma})$ where $\hat{\sigma}$ is the whole set of constant laminate conductivities.

Briane and Milton showed that there is a set $\hat{P}$ of constant $d \times d$ matrices such that $P_\varepsilon = L_n^\varepsilon(\hat{P})$ is a corrector (or a matrix electric field) associated to the conductivity $\sigma_\varepsilon = L_n^\varepsilon(\hat{\sigma})$ in the sense of Murat-Tartar:

$$\begin{cases} P_\varepsilon \rightharpoonup I & \text{weakly in } \mathcal{L}^2_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R}^{d\times d}), \\ \mathrm{Curl}(P_\varepsilon) \to 0 & \text{strongly in } \mathcal{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R}^{d\times d}), \\ \mathrm{Div}(\sigma_\varepsilon P_\varepsilon) & \text{is compact in } \mathcal{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R}^d). \end{cases} \tag{6}$$

The weak limit of $\sigma_\varepsilon P_\varepsilon$ in $\mathcal{L}^2_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{R}^{d\times d})$ is then the homogenized limiting conductivity of the laminate. The three conditions (6) satisfied by P_ε extend in the laminate case to the three respective conditions

$$\begin{cases} \langle DU \rangle = I, \\ \mathrm{Curl}(DU) = 0, \\ \mathrm{Div}(\sigma DU) = 0. \end{cases} \tag{7}$$

satisfied by any electric field DU in the periodic case.

Theorem ($\mathbb{L}$ 1)

Let $d, n \in \mathbf{N}$. Consider the rank-n laminate multiscale field $L_n^\varepsilon(\hat{P})$ built from a finite set $\hat{P}$ of $d \times d$ matrices satisfying

$$\begin{aligned} P_\varepsilon \rightharpoonup I \quad & \text{weakly in } \mathcal{L}^2_{loc}(\mathbf{R}^d, \mathbf{R}^{d\times d}), \\ \operatorname{Curl}(P_\varepsilon) \to 0 \quad & \text{strongly in } \mathcal{H}^{-1}_{loc}(\mathbf{R}^d, \mathbf{R}^{d\times d}), \end{aligned} \tag{8}$$

Then necessary and sufficient that the field be realized, i.e., $\operatorname{Div}(\sigma_\varepsilon P_\varepsilon)$ is compact in $\mathcal{H}^{-1}_{loc}(\mathbf{R}^d, \mathbf{R}^d)$ for some rank-n laminate conductivity $L_k^\varepsilon(\hat{\sigma})$ is that $\det\big(L_k^\varepsilon(\hat{P})\big) > 0$ a.e. in $\mathbf{R}^d$, or equivalently, that the determinant of each matrix in $\hat{P}$ is positive.

Determinant positivity follows from a theorem of Briane, Milton and Nesi. Conversely, suppose there is a laminate field $P_\varepsilon = L_n^\varepsilon(\hat{p})$ satisfying (8) and $\det(P_\varepsilon) > 0$ *a.e.*

Pf. Torus

As in the matrix field case consider the rank-n conductivity defined by

$$\sigma_\varepsilon = \det(P_\varepsilon)\,(P_\varepsilon^{-1})^T\,P_\varepsilon^{-1} = L_n^\varepsilon(\hat\sigma),$$

where $\hat\sigma = \{\det(P)\,(P^{-1})^T\,P^{-1} : P \in \hat P\}$. Then compactness is equivalent to the compactness of

$$\mathrm{Div}(\mathrm{Cof}(P_\varepsilon)).$$

Contrary to the periodic case, $\mathrm{Cof}(P_\varepsilon)$ is not divergence free as a distribution. But using the homogenization procedure for laminates of Briane, by the quasi-affinity of cofactors for gradients compactness holds if the matrices P and Q of two neighboring layers in a direction ξ of the laminate satisfy the jump condition for the divergence

$$\big(\mathrm{Cof}(P) - \mathrm{Cof}(Q)\big)^T\xi = 0. \tag{9}$$

More precisely, at the given scale ε_k of the laminate the matrix P or Q is either a matrix in $\hat{P}$ or the average of rank-one laminates obtained at the smallest scales $\varepsilon_{k+1}, \ldots, \varepsilon_n$

In the first case the matrix P is the constant value of the field in a homogeneous layer of the rank-n laminate.

In the second case, the average of the cofactors of the matrices involved in those rank one laminations is equal to the cofactors matrix of the average, $\mathrm{Cof}(P)$, by virtue of the quasi-affinity of the cofactors applied iteratively to the rank-one connected matrices in the rank-one laminate.

Therefore, it remains to prove (9) for any matrices P and Q with positive determinant satisfying the condition that controls the jumps in the convergence of $\mathrm{Curl}(P_\varepsilon) \to 0$ in (8).

For any matrices P and Q with positive determinant satisfying the condition we must show

$$P - Q = \xi \otimes \eta, \qquad \text{for some } \eta \in \mathbf{R}^d.$$

So by multiplicativity of cofactor matrices we have

$$\begin{aligned}\big(\mathrm{Cof}(P) - \mathrm{Cof}(Q)\big)^T \xi &= \mathrm{Cof}(Q)^T \left[\mathrm{Cof}\big(I + (\xi \otimes \eta) Q^{-1}\big)^T - I\right] \\ &= \mathrm{Cof}(Q)^T \left[\mathrm{Cof}(I + \xi \otimes \lambda)^T - I\right]\end{aligned}$$

where $\lambda = (Q^{-1})^T \eta$. Moreover if $\xi \bullet \eta \neq -1$ we have

$$\mathrm{Cof}(I + \xi \otimes \lambda)^T = \det(I + \xi \otimes \lambda)\,(I + \xi \otimes \lambda)^{-1} = (I + \xi \bullet \lambda) I - \xi \otimes \lambda,$$

which extends to the case $\xi \bullet \eta = -1$ by continuity. Hence

$$\big(\mathrm{Cof}(P) - \mathrm{Cof}(Q)\big)^T = \mathrm{Cof}(Q)\big((\xi \bullet \lambda) I - \xi \otimes \lambda\big),$$

which implies (9) since $(\xi \otimes \lambda)\xi = (\xi \bullet \lambda)\xi$. □

Thanks!